$$\frac{\partial^m}{\partial y^m}\left(\frac{e^{-ikr_0}}{r_0}\right) = (-ik)^m \left(\frac{y}{r_0}\right)^m \frac{e^{-ikr_0}}{r_0} + 0\left(r_0^{-2}\right),$$

$$\frac{\partial}{\partial z}\frac{\partial^{m-1}}{\partial y^{m-1}}\left(\frac{e^{-ikr_0}}{r_0}\right) = (-ik)^m \frac{z}{r_0}\left(\frac{y}{r_0}\right)^{m-1} \frac{e^{-ikr_0}}{r_0} + 0\left(r_0^{-2}\right),$$

$$z/r_0 = \sqrt{1-l_x^2}\sin\theta,\quad y/r_0 = \sqrt{1-l_x^2}\cos\theta.$$

Taking these expressions into account, (21) and (25) for the amplitude functions Φ_1, Φ_2 can be represented for $r_0 \gg \lambda$ in the form

$$\Phi_1(x, y, z) = \frac{e^{-ikr_0}}{4\pi r_0}\sum_{m=0}^{n}\left(-ik\sqrt{1-l_x^2}\cos\theta\right)^m \int_{-\lambda}^{+\lambda} Q_{1m}(\xi)\, e^{ikl_x\xi} d\xi + 0\left(r_0^{-2}\right); \tag{31}$$

$$\Phi_2(x, y, z) = \frac{e^{-ikr_0}}{4\pi r_0}\sin\theta \sum_{m=1}^{n}\left(-ik\sqrt{1-l_x^2}\right)^m \cos^{m-1}\theta \int_{-\lambda}^{+\lambda} Q_{2m}(\xi)\, e^{ikl_x\xi} d\xi + 0\left(r_0^{-2}\right). \tag{32}$$

It follows from (31) and (32) that in the case of low-frequency oscillations of a body, when the parameter is $k = \omega R_0/a \ll 1$, the main contribution to the sound field far from the oscillating body is due to its axisymmetric oscillations.

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SHOCK LOADING OF AN INFINITE PLATE CONTIGUOUS TO A FLUID

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Questions of the effect of shock loadings on infinite plates in contact with a fluid have been examined in a number of papers [1-9]. The axisymmetric deformation of plates was studied in [1-6], while [1, 7-9] were devoted to the plane problem. The investigations were executed in different formulations. Different kinds of plate loadings (the effect of acoustic pressure waves, concentrated forces or distributed loads; assignment of the motion velocity) were considered. The plate deformation was described by different equations (the membrane deflection equation, the Bernoulli–Euler bending equation, or a Timoshenko-type equation). The main method of solving these problems is the method of integral transforms. Definite difficulties occur during the solution in going from the transforms to the originals. Still greater difficulties are encountered in analyzing the solution and obtaining specific numerical results in the originals written in the form of complex single or double integrals. The solution in a number of papers [3, 5, 8] is hence constrained to the writing of formulas in quadratures, while the problem is solved in other investigations [1, 2, 4, 6, 9] by asymptotic methods which are valid in a definite range of time variation. There are also separate results obtained by using the numerical inversion of the Laplace transform in [6] which is devoted to the effect of a spherical pressure wave.

In this paper, the solution of the plane problem of bending an infinite plate in contact with a compressible fluid occupying a half-space along one of the sides of the plate is sought by using integral transforms.

§1. The X, Z coordinate plane is in the plane of the plate, and the Y axis is directed into the fluid. A transverse load distributed uniformly along the Z axis is applied instantaneously to the plate along this whole axis. It is sufficient to consider the motion in one

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plane of the X, Y variables. The state of the fluid in this plane is described by a two-dimensional wave equation and the plate deformation is described by the Bernoulli–Euler equation for the bending of a beam strip to which a lumped force has been applied in the section X = 0. A solution is later presented which takes account of shear strain of the plate in addition to the bending.

Dimensionless quantities are used and the following notation is introduced:

$$t = Tc\delta^{-1},\ x = X\delta^{-1},\ y = Y\delta^{-1},\ \varphi = \Phi c^{-1}\delta^{-1},\ u = U\delta^{-1},\ v = Vc^{-1},$$

where δ is the plate thickness; T is the time; c is the speed of sound in the fluid; c_1 is the speed of sound in the plate material; ρ, ρ_1 are the densities of the fluid and the plate material; Q is a constant with the dimensionality of a linear load; Φ is the fluid velocity potential; U,V are the transverse displacement and velocity of the plate; h is the unit Heaviside function; and p, q are the Laplace and Fourier transform variables.

The equations of state of the fluid and of beam-strip bending in dimensionless variables have the form

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \frac{\partial^2\varphi}{\partial t^2},\quad a\frac{\partial^4 u}{\partial x^4} + \frac{\rho_1\partial^2 u}{\rho\partial t^2} = \frac{\partial\varphi}{\partial t}\quad (y = 0), \tag{1.1}$$

where

$$a = \frac{1}{12}\frac{\rho_1 c_1^2}{\rho c^2}.$$

Zero initial and the following boundary conditions are given:

$$\frac{\partial\varphi}{\partial y} = \frac{\partial u}{\partial t}\quad (y = 0),\quad \varphi \neq \infty\quad (y \to \infty),\quad \frac{\partial\varphi}{\partial x} = 0\quad (x = 0); \tag{1.2}$$

$$\frac{\partial u}{\partial x} = 0,\quad \frac{\partial^3 u}{\partial x^3} = \frac{6Q}{\rho_1 c_1^2\delta}h(t)\quad (x = 0),\quad u \to \infty\quad (x \to \infty). \tag{1.3}$$

A Laplace transform in the variable t and a Fourier cosine transform in the variable x are applied to (1.1). Taking account of the zero initial and corresponding boundary conditions for the transforms, we obtain the equations

$$\frac{\partial^2\varphi^{FL}}{\partial y^2} - (p^2 + q^2)\,\varphi^{FL} = 0; \tag{1.4}$$

$$a\left(\frac{6Q}{\rho c_1^2\delta}p^{-1} + q^4 u^{FL}\right) + \frac{\rho_1}{\rho}p^2 u^{FL} = p\varphi^{FL}\quad (y = 0). \tag{1.5}$$

The letter superscripts F and L denote, respectively, the Fourier and Laplace transforms. Therefore,

$$u^{FL} = \int\limits_0^\infty \cos(qx)\int\limits_0^\infty \exp(-pt)\,u(t, x)\,dtdq.$$

The solution of the differential equation (1.4) for the variable y satisfying the first two conditions (1.2) (written in the appropriate transforms) has the form

$$\varphi^{FL} = -\frac{pu^{FL}}{\sqrt{p^2 + q^2}}\exp\left(-y\sqrt{p^2 + q^2}\right).$$

Substituting this value into (1.5), the transform u^{FL} can be found. Henceforth, the plate velocity will be considered. The dimensionless quantity v corresponding to the velocity has the transform

$$v^{FL} = pu^{FL} = \frac{Q}{2\delta\rho c^2}\left(aq^4 + \frac{\rho_1}{\rho}p^2 + \frac{p^2}{\sqrt{p^2 + q^2}}\right)^{-1}.$$

By using identity transformations this expression is reduced to the form

$$v^{FL} = \frac{Q}{2\delta\rho c^2}\frac{\rho}{\rho_1}\frac{1}{q\sqrt{p^2 + q^2}}f^L(s),$$

$$f^L(s) = \frac{s^2}{s^3 + Bs^2 + Cs - B},\quad s = \sqrt{p^2q^{-2} + 1}, \tag{1.6}$$

$$B = \frac{\rho}{q\rho_1},\quad c = \frac{1}{12}\frac{c_1^2}{c^2}q^2 - 1.$$

The following Laplace transform formula, obtained from [10], is used. If $f^L(s) = L[f(t)]$, then the following holds:

$$\frac{1}{q\sqrt{p^2+q^2}} f^L\left(\sqrt{p^2q^{-2}+1}\right) = L\left[\int_0^t J_0\left(q\sqrt{t^2-\xi^2}\right) f(q\xi)\, d\xi\right], \tag{1.7}$$

where L is the Laplace transform symbol [11]; s is the new notation for the Laplace transform variable; J_0 is the zero-order Bessel function. Writing the transform in the form (1.6) corresponds to the left side in (1.7).

The function $f^L(s)$ in (1.6) can be considered as a rational fraction in the new Laplace transform variable s. The original of such a function is found simply [11]. Using this original and applying (1.7), we perform a passage to the original (Laplace) from the transform (1.6). Applying the Fourier transform inversion formula to the result obtained, we write the final expression for the plate velocity in the form

$$V = cv = \frac{1}{\pi}\frac{Q}{\rho c\delta}\frac{\rho}{\rho_1}\int_0^\infty \cos(qx)\int_0^t J_0\left(q\sqrt{t^2-\xi^2}\right)\left[\sum_{k=1}^{3}\frac{s_k^2\exp(s_k q\xi)}{R'(s_k)}\right] d\xi dq, \tag{1.8}$$

where $R'(s_k) = 3s_k^2 + 2Bs_k + C$, s_k (k = 1, 2, 3) are roots of the cubic equation $R(s) = s^3 + Bs^2 + Cs - B = 0$.

Formula (1.8) remains valid when taking account of the shear strain if the coefficient C therein is taken in the form

$$C = qc_1^2c^{-2}\left(q^2c_1^2c_2^{-2}+12\right)^{-1} - 1, \quad c_2 = \sqrt{0.833G\rho_1^{-1}},$$

where G is the shear modulus of the plate material.

§2. The roots s_k (k = 1, 2, 3) of the cubic equation are functions of the variable q. One root among them is real and positive for any values of q while the other two are either real and negative or complex conjugates with negative real parts. The exponential term in the integrand of (1.8), which increases rapidly with the increase in ξ, corresponds to the positive root. The product of this term by the Bessel function yields a function which osciliates at high frequency relative to zero and with abruptly growing amplitudes. Computation of the integral (1.8) by numerical methods from this function is difficult.

The integrand is converted by means of the formula

$$\exp(s_1q\xi) = 2\,\mathrm{ch}\,(s_1q\xi) - \exp(-s_1q\xi),$$

where s_1 is the positive root of the cubic equation. This value is substituted into (1.8). On the basis of the expression

$$\int_0^t J_0\left(q\sqrt{t^2-\xi^2}\right)\mathrm{ch}\,(s_1q\xi)\, d\xi = \frac{\sin\left(qt\sqrt{1-s_1^2}\right)}{q\sqrt{1-s_1^2}},$$

resulting from (6.677) in [12] for an imaginary value of the argument in the cosine, we may obtain

$$V = \frac{1}{\pi}\frac{Q}{\rho c\delta}\frac{\rho}{\rho_1}\int_0^\infty \cos(qx)\left(\frac{2s_1^2}{R'(s_1)}\frac{\sin\left(qt\sqrt{1-s_1^2}\right)}{q\sqrt{1-s_1^2}} + F\right) dq, \tag{2.1}$$

where F is the integral whose integrand contains exponential terms with arguments with negative real parts. As numerical computations have shown, the contribution of the term F in (2.1) is small in a wide range of t and diminishes with the growth in t. This term can be neglected for a steel plate and water and t > 10. In the case of an incompressible fluid (c → ∞), formula (2.1) takes the form

$$V = \frac{1}{\pi}\frac{Q}{\rho_1c_1\delta}\int_0^\infty q^{-2}\sqrt{12q\left(q+\rho\rho_1^{-1}\right)}^{\,-1}\sin\left(t_1q^2\sqrt{bq}\right)\cos(qx)\, dq,$$

$$t_1 = Tc_1\delta^{-1}, \quad b = \left[12\left(q+\rho\rho_1^{-1}\right)\right]^{-1}.$$

According to [1], the asymptotic formula

$$V = \frac{2}{5\pi}\frac{Q}{\rho c\delta}t^{1/5}a^{-2/5}\int_0^\infty \sin y\cos\left(a^{-1/5}y^{2/5}\varkappa\right)y^{-6/5}dy, \quad \varkappa = xt^{-2/5} \tag{2.2}$$

can be obtained for large values of the time. For x = 0 the integral in (2.2) is expressed in terms of the Γ-function [12]:

$$\int_0^\infty y^{-6/5} \sin y dy = \Gamma(-1/5) \sin(-\pi/10) \approx 1.8.$$

Hence, the asymptotic formula becomes for x = 0

$$V = 0.229 \frac{Q}{\rho c \delta} t^{1/5} a^{-2/5}. \tag{2.3}$$

It is sometimes assumed in performing the approximate computations that the pressure acting from the fluid on the body inserted therein equals the product of the acoustic drag ρc by the normal component of the velocity of the point under consideration on the body (plane reflection hypothesis). To obtain a solution by this hypothesis, the value of the appropriate pressure is substituted into the right side of the second equation in (1.1). Then the Laplace transform is applied to the equation. An ordinary differential equation in the variable x is hence obtained. This equation is solved for appropriate boundary conditions, and a Laplace transform of the quantity v is found in the section X = 0,

$$v^L = 0.66 \frac{Q}{\rho_1 c \sqrt{cc_1} \delta} (p^2 + \rho \rho_1^{-1} p)^{-3/4}.$$

After going over to the original, a formula can be obtained for the plate velocity,

$$V = cv = 1.13 \frac{Q}{\rho c \delta} \sqrt{\frac{\rho c}{\rho_1 c_1}} \tau^{1/4} \exp(-\tau) I_{1/4}(\tau), \tag{2.4}$$

where $I_{1/4}$ is the Bessel function of imaginary argument of order ¼; $\tau = 0.5 \rho \rho_1^{-1} t$.

§3. Computations using the formulas obtained were performed on the "Promin'" and "Mir" digital computers. The quantity $r = 1 - s_1$, for which the appropriate cubic equation was formed, was determined in the computer program instead of the root s_1. This permitted avoidance of the calculation of small differences during computation of the quantities in (2.1).

As $q \to 0$, the root s_1 tends to one and the integrand has an indeterminacy. Hence, the series expansion of the integrand in the neighborhood of the point q = 0 was used, which permitted resolution of the indeterminacy and execution of the integration for small values of the variable q.

Results of computations for a steel plate contiguous to water ($\rho_1/\rho = 7.85$) are presented below.

A graph of the plate velocity in the section X = 0 (curve 1) is presented in Fig. 1; curve 2 has been obtained under the assumption that the fluid is incompressible. The results of both computations turn out to be close in the whole range of values of t except the origin (t < 5). Neglecting the fluid compressibility results in an error in determining the velocity of 16% for t = 10, 4% for t = 50, and 2% for t = 250, respectively, which is completely acceptable for execution of the computations.

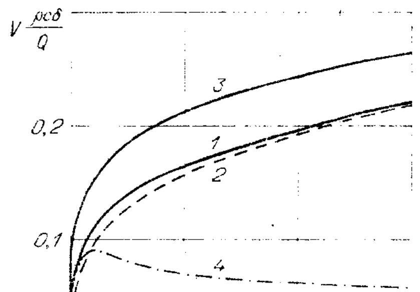

Fig. 1

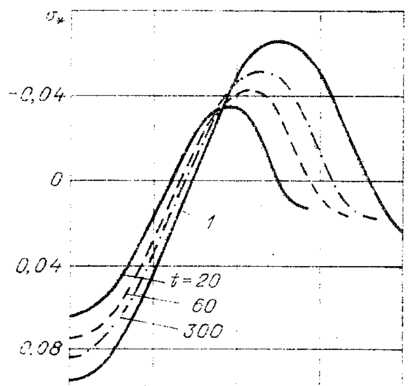

Fig. 2

Shear strain appears only in the initial moment of plate motion, and its influence then drops rapidly. Computations executed for a steel plate for $t = 20$ with and without shear taken into account agree in practice. This result agrees with the deduction in [1] that taking account of shear can be required only at times from the beginning of load action which are commensurate with several periods of longitudinal wave traversal over the plate thickness (to four periods).

For large values of the time the asymptotic formulas (2.2) and (2.3) correspond to the solution in which fluid compressibility and plate inertia are neglected [1]. This means that the influence of the factors mentioned are asymptotically inessential. A graph of the asymptotic value of the velocity is superposed in Fig. 1 (curve 3), from which it follows that for $t > 100$ the asymptotic solution yields a relative error less than 25% which diminishes with the rise in t.

Shown in Fig. 2 are velocity distributions along the plate length, computed by using the asymptotic and exact methods. The graphs are constructed in the coordinate axes $v_* = VQ^{-1}\rho c\delta t^{-1/5}$, $\varkappa$. Curve 1 in the coordinate plate of the variables v_*, $\varkappa$ which corresponds to the asymptotic solution, occupies a fixed position. The remaining curves corresponds to values of the velocity obtained by the exact method for different values of t, and approach the curve 1 as t grows. The best agreement between the asymptotic and exact solutions is observed for bounded values of $\varkappa$. The first intersections of the curves with the $v_* = 0$ axis (the first nodes) are close to each other in a broad range of variation in the variable t. Closure for the second points of intersection is observed for $t > 300$.

The graph of the velocity (curve 4) computed by (2.4) in conformity with the plane reflection hypothesis in Fig. 1 agrees with the exact solution for bounded t and yields a satisfactory result up to $t < 12$ (with an error less than 25%).

Estimating the possibility of applying the plane reflection hypothesis to the solution of different problems, it should be kept in mind that this hypothesis corresponds to constrained conditions of fluid motion in which the influence of fluid compressibility turns out to be essential.

The plate dimensions and fluid volume are not constrained in the specific problem under consideration, and the load is applied at one section of the plate. Under such conditions, the fluid has great freedom of displacement along the plate surface. The fluid presses on sections of the plate far from the site of load application, opposite to the direction of action of the force. The fluid itself is hence squeezed insignificantly.

If the acting loads are represented in the form of a certain set of forces distributed over the plate surface, or the plate and fluid are bounded in size, then conditions constraining the spread of the fluid along the plate surface can be produced. Under these conditions, taking account of fluid compressibility can be required in a broader range of times and it generally turns out to be impossible to accomplish displacement for certain kinds of motion without taking account of fluid compressibility.

Similar conditions are produced, for instance, under the effect of a uniformly distributed transverse load applied over the whole surface of an infinite plate. In this case, the plane reflection hypothesis yields the exact result.

The reasoning expressed indicates that selection of an acceptable computational hypothesis to take account of the influence of the fluid in each specific problem requires careful consideration.

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A SELF-SIMILAR SOLUTION FOR FAN JETS WITH AN ARBITRARY DEGREE OF SWIRLING

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Solutions are known [1-4] to the problem of the propagation of swirled fan (radial) jets into a submerged space. Functions which are valid at a distance considerably exceeding the radius of the round slit, where the jet is always weakly swirled, are obtained in [1, 2]. In the search for a solution for a jet discharging from an infinitely narrow slit of finite radius the assumption of weak swirling of the jet was introduced in [3] as an auxiliary assumption. In [4], where several terms of an asymptotic expansion by inverse powers of the distance from the nozzle were found for a laminar jet with a considerable swirling, the question of the determination of the integration constants remains open.

In the present paper it is shown that the problem of the propagation of a fan jet discharging from an infinitely narrow slit of finite radius has a self-similar solution for any degree of swirling of the jet.

§1. In the approximation of boundary-layer theory the equations describing the flow in swirled laminar or turbulent fan jets of incompressible liquid have the following form in the cylindrical coordinate system x, y, φ (the x axis is directed perpendicular to the axis of symmetry and φ is the polar angle)

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\frac{w^2}{x}=\frac{1}{\rho}\frac{\partial \tau_x}{\partial y}; \tag{1.1}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{uw}{x}=\frac{1}{\rho}\frac{\partial \tau_\varphi}{\partial y}; \tag{1.2}$$

$$\frac{\partial (xu)}{\partial x}+\frac{\partial (xv)}{\partial y}=0, \tag{1.3}$$

where u, v, and w are the components of the velocity vector in the directions of the x, y, and φ axes; τ_x and τ_φ are the components of the shear stress of friction in the directions of the x and φ axes; ρ is the density of the liquid.

First let us consider a free submerged jet. Then the system (1.1)-(1.3) must be integrated with the following boundary conditions:

$$\begin{cases} u=0, \quad w=0 & \text{at} \quad y=\pm\infty; \\ \dfrac{\partial u}{\partial y}=0 & \text{at} \quad y=0. \end{cases} \tag{1.4}$$

The goal of the present report is to find a self-similar solution, and therefore the initial condition loses its importance. The two integral conditions of conservation needed for complete determinacy of the problem will be obtained in the course of the solution.

We will adopt the widely prevalent hypothesis that the following relationship is valid not only for laminar flows but also for turbulent flows of the boundary-layer type:

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